Precursory dynamics in threshold systems

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(Received 22 January 2001; revised manuscript received 21 February 2002; published 17 May 2002)

A precursory dynamics, motivated by the analysis of recent experiments on solid-on-solid friction, is introduced in a continuous cellular automaton that mimics the physics of earthquake source processes. The resulting system of equations for the interevent cycle can be decoupled and yields an analytical solution in the mean-field limit, exhibiting a smoothing effect of the dynamics on the stress field. Simulation results show the resulting departure from scaling at the large-event end of the frequency distribution, and support claims that the field leakage may parametrize the superposition of scaling and characteristic regimes observed in real earthquake faults.

DOI: 10.1103/PhysRevE.65.056117 PACS number(s): 64.60.Fr, 64.60.My, 64.70.Pf, 05.70.Jk

There are many examples in nature of systems whose internal dynamics have as an essential feature the separation of time scales into a long-term loading process and a short-term discharge. In the former, some external source slowly drives the variables that control the dynamics until these fields reach some threshold value. When this happens, the fast dynamics of the discharge takes over and generates a sequence of internal rearrangements, called an avalanche in this context, which eventually removes the excess through internal dissipation, or by flow across the system's borders. These dynamical threshold systems have been the object of much recent interest, due mainly to the fact that they can be recast into a discrete time-evolution form. The resulting discrete dynamics transforms the fast time scale discharge cycle into a sequence of well-defined time steps, in each of which the dynamical fields undergo a stochastic transformation that depends on their configuration in the previous step as well as annealed noise. As such, and since the fields are in general real valued, we will call these representations stochastic continuous cellular automata (SCCA).

Physical systems that have been modeled and studied along these lines are abundant in the literature, ranging from the firing behavior of neural networks [1] to the dynamics of domain walls in magnets [2], the motion of vortex lines in type II superconductors [3], the depinning transition in the growth of interfaces in random media [4], the dynamics of the energy release in solar flares [5], and to the source processes responsible for earthquakes.

The existence of an interevent dynamical cycle in the latter is suggested by recent laboratory experiments addressing issues of solid-on-solid friction. A stable slip, with a slow velocity that increases with the stress level, is observed prior to failure, leading to a partial release of the accumulated stress [6,7]. This stress leakage mechanism is analogous to a temperature-dependent viscosity that has been observed in laboratory for the creeping of crystalline rocks, and can be modeled by the equation

$$\frac{ds(t)}{dt} = \alpha \frac{\sigma(t) - \sigma_R}{K}, \tag{1}$$

where $s(t)$ and $\sigma(t)$ are the displacement and the stress at time t, σ_R is some residual stress value to which the system decays after $\sigma(t)$ reaches a failure threshold σ_F, K is the elastic stiffness, and α measures the intensity of this stress leakage effect. This parameter, with dimensions of inverse time, will in general be dependent on both the stress level and the temperature. An analogous leakage mechanism has also been suggested in the context of integrate-and-fire neural networks [1], and it is likely that the results reported here will also hold in that context.

As a summary, we will show that the introduction of a stress leakage process as an interevent dynamics in a SCCA model for earthquake faults changes in a dramatic way the space-time patterns it generates. In particular, the α value for a fault may determine its overall behavior as of a scale-invariant type or a nucleation type, with a mixed composition in between, reproducing features observed in real faults [8]. The importance of this new parameter in earthquake source models has in fact been recently evaluated. Its tuning to match the characteristics of each segment in a complex computer representation of the fault network of southern California allowed the generation of space-time patterns of rupture of unprecedented realism [9].

We will consider here the Rundle-Jackson-Brown (RJB) SCCA model for an earthquake fault [10], in its uniform long-range interaction, i.e., the mean-field version [11]. Extensive work has recently focused a near-mean-field version of this model, where the interaction range has a cutoff and the model can be mapped onto an Ising-like Langevin equation [12]. Its dynamical variables are two continuous real-valued fields, slip $s_i(t)$ and stress $\sigma_i(t)$, defined on the sites i of a lattice. A constitutive equation couples these fields,

$$\sigma_i(t) = \sum_j T_{ij} s_j(t) + K_L v t, \tag{2}$$

where T_{ij} is the interaction matrix, or stress Green's function, $T_{ii} = -K$, $K = K_L + K_C$, $K_C = \Sigma_{j \neq i} T_{ij}$, K_L parametrizes the loading interaction, and v fixes a relation between

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the short and long time scales of the physics of the model. Equation (2) specifies the stress at each site as long as $\max[\sigma_i(t)]<\sigma_F$, which defines the stress threshold σ_F, taken as uniform over the lattice. The interevent, or loading, dynamics of the standard RJB model is thus very simple; the slip field remains static while the stress field undergoes linear growth. As soon as the stress at one site reaches the threshold, a fast stochastic relaxation dynamics takes over. The slip at a site that fails is discontinuously reset to a new value that leads the stress to assume its residual value, usually with some noise, introduced to represent the disorder in the rheology. The stress drop is redistributed among the interacting neighbors, with an intrinsic dissipation measured by the factor $\delta=K_L/K$:

$$\Delta\sigma_j=\frac{T_{ji}}{K}|\Delta\sigma_i|, \tag{3}$$

where i is the site that failed. This increase may cause other sites to fail as well, and the process continues until all sites have stress below failure. This cascade of failures, or avalanche, is the model's equivalent for an earthquake.

The relaxation dynamics of the model can be cast into a single field formulation that is specially convenient for computer simulations. In units of the short time scale,

$$\sigma_i(t+1)=\sigma_i(t)+\sum_j\left\{T_{ij}\frac{\sigma_j(t)-\sigma_R}{K}\Theta(\sigma_j(t)-\sigma_F)\right\}+\xi_i(t), \tag{4}$$

where $\xi_i(t)$ is the noise term. We direct the reader to Ref. [11] for a complete and pedagogical discussion of this relaxation dynamics.

We report in this paper results obtained for the introduction of stress leakage, as modeled by Eq. (1), as an interevent dynamics of our SCCA model. The equations for these interevent dynamics are obtained by taking the time derivative of Eq. (2), substituting $ds_i(t)/dt$ from Eq. (1), and choosing, with no loss of generality, $\sigma_R=0$, to get,

$$\frac{d\sigma_i(t)}{dt}=\frac{\alpha}{K}\sum_j T_{ij}\sigma_j(t)+K_L v, \tag{5}$$

a set of N coupled equations for the stress field. Without introducing any particular form for the T_{ij}, these equations can be combined to derive the time evolution of the average stress for short times $\overline{\sigma(t)}=\Sigma_i\sigma_i(t)/N$:

$$\frac{d\overline{\sigma}(t)}{dt}=-\alpha\frac{K_L}{K}\overline{\sigma}(t)+K_L v, \tag{6}$$

with the solution, cast in dimensionless form by defining $\tilde{t}=\sigma_F/K_L v$, $\eta=\sigma/\sigma_F$, $\tau=t/\tilde{t}$, and $\phi=\alpha\tilde{t}$,

$$\overline{\eta}(\tau)=\left(\overline{\eta}(0)-\frac{1}{\delta\phi}\right)\exp(-\delta\phi\tau)+\frac{1}{\delta\phi}, \tag{7}$$

which is an increasing (decreasing) function of dimensionless time if $\overline{\eta}(0)<(>)1/\delta\phi$.

Equations (4) and (5) above define the generic RJB model with stress leakage for interevent dynamics. The current focus of research is on the long-range versions, where the interaction matrix T_{ij} is nonzero for a substantial fraction of all pairs (ij). Ideal elasticity indicates that this matrix should involve a dependence on the distance as $1/r^3$, as expected from the results of simple laboratory experiments. Nevertheless, it has been shown that the upper critical dimension for this interaction is $d_u=2$ [13] and we can recover the long wavelength physics of this model by a simpler mean-field formulation. We will thus focus on the mean-field RJB model, with a uniform interaction matrix $T_{ij}=K_c/(N-1)$, where $N=1/\Delta$ is the number of sites in the lattice. In this case, the set of equations in Eq. (5) can be decoupled and yield a solution that reads, in dimensionless variables,

$$\eta_i(\tau)=[\eta_i(0)-\overline{\eta}(0)]\exp\left\{-\left[\frac{1-\delta\Delta}{1-\Delta}\right]\phi\tau\right\}+\left(\overline{\eta}(0)-\frac{1}{\delta\phi}\right)\exp(-\delta\phi\tau)+\frac{1}{\delta\phi}. \tag{8}$$

From

$$\eta_i(\tau)-\eta_j(\tau)=[\eta_i(0)-\eta_j(0)]\exp\left\{-\left[\frac{1-\delta\Delta}{1-\Delta}\right]\phi\tau\right\},$$

we can see that this time evolution of the stress field is order preserving, i.e., $\eta_i(0)>\eta_j(0)\Rightarrow\eta_i(t)>\eta_j(t)$, allowing for the determination of the initiator of the next event by finding the site with maximum stress at the completion of the previous event. The important effect of this leakage stress dynamics in the pattern of failures of the system comes from the reduction it causes in the statistical spread of the stress field with time. A simple measure of this smoothing is obtained through the time evolution of the variance of the stress field

$$\mathrm{var}[\eta(t)]=\mathrm{var}[\eta(0)]\exp\left\{-\left[\frac{1-\delta\Delta}{1-\Delta}\right]\phi\tau\right\}. \tag{9}$$

Because of this exponential smoothing of the stress field, the probability of a site to reach failure after receiving a transfer from a failing neighbor increases, and is an increasing function of the time to failure. As a consequence, the branching ratio, defined as the average number of failures caused by each failing site, also increases. The system is more likely to undergo larger avalanches, which may even be system wide when the time to failure is large enough.

Because this interevent dynamics preserves order, as already mentioned, its introduction in a SCCA is rather straightforward. The site with stress nearest to failure η_{max} after an event is the initiator of the next, and solving for τ in Eq. (8) for this site determines the time-to-failure τ_F that will be used again to update the stress field over the lattice prior to the event. To avoid the time-consuming solving of transcendental equations at each time step in the simulation, the

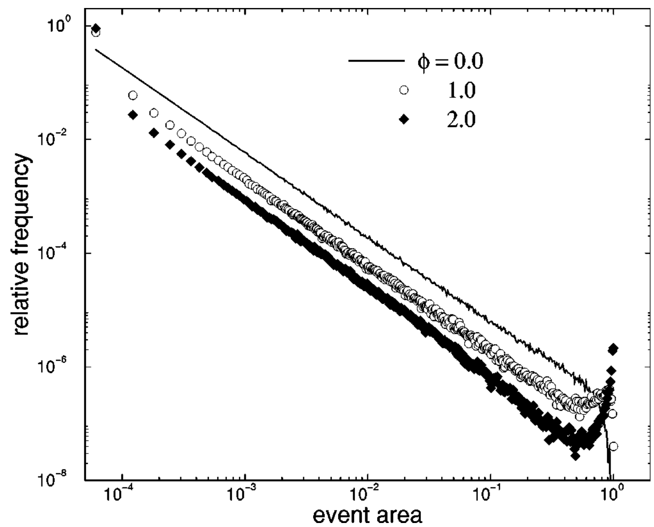

FIG. 1. The frequency distribution is shown for dimensionless leakage factors $\phi = 1.0$ and 2.0; the plot with no leakage is shown for a comparison. The simulations were performed on a 128×128 lattice, with a dissipation factor $\delta = 0.01$ and a noise amplitude of 0.5. Data for this and all subsequent plots were collected from 3 000 000 events, after a transient of the same order, and are logarithmically binned.

time-evolution equations may be linearized, as long as $\max\{\delta\phi\tau, [(1-\delta\Delta)/(1-\Delta)]\phi\tau\} \ll 1$, which is true for our parameters, to read

$$\eta_i(\tau) = \eta_i(0) + \left\{ 1 + \left[\frac{1-\delta}{1-\Delta}\right] \phi\, \overline{\eta}(0) - \left[\frac{1-\delta\Delta}{1-\Delta}\right] \phi\, \eta_i(0) \right\} \tau,$$

and the solution for the time to failure is

$$\tau_F = \frac{1-\eta_{max}}{1 + \left[\frac{1-\delta}{1-\Delta}\right] \phi\, \overline{\eta}(0) - \left[\frac{1-\delta\Delta}{1-\Delta}\right] \phi\, \eta_{max}}. \qquad (10)$$

Figure (1) shows a log-log plot of the frequency distribution of events as a function of their size. The effect of the stress leakage dynamics shows up clearly in the excess over scaling obtained for large events as ϕ is increased from 0, together with a depletion of the distribution in the intermediate size range. The slope of the scaling part of the plot also gradually increases, starting from the mean-field value $\tau = 1.5$. The smoothing effect of the leakage dynamics, together with the resulting larger stress average that it causes in the field as a whole, results in a higher probability for large events to grow, eventually causing total rupture of the fault [15].

This smoothing effect is reflected also in the distribution for the number of topplings, shown in the lower plot of Fig. 2. Here, a counter is updated each time a site fails, even if it had failed before in the same avalanche. This number reflects more closely the model's equivalent for the moment release in an event. The fraction of multiple failures vanishes in the exact mean-field limit, and the number of topplings become equivalent to the event area. This is no longer true for the model with stress leakage. The single maximum at the upper end of Fig. 1 corresponds, in fact, to very different moment releases, as shown in Fig. 2. An intriguing feature of the latter is the double bump at the large end, located close to integer multiples of the system size. They suggest that total rupture of the system is more likely, in this region, than one would naively expect.

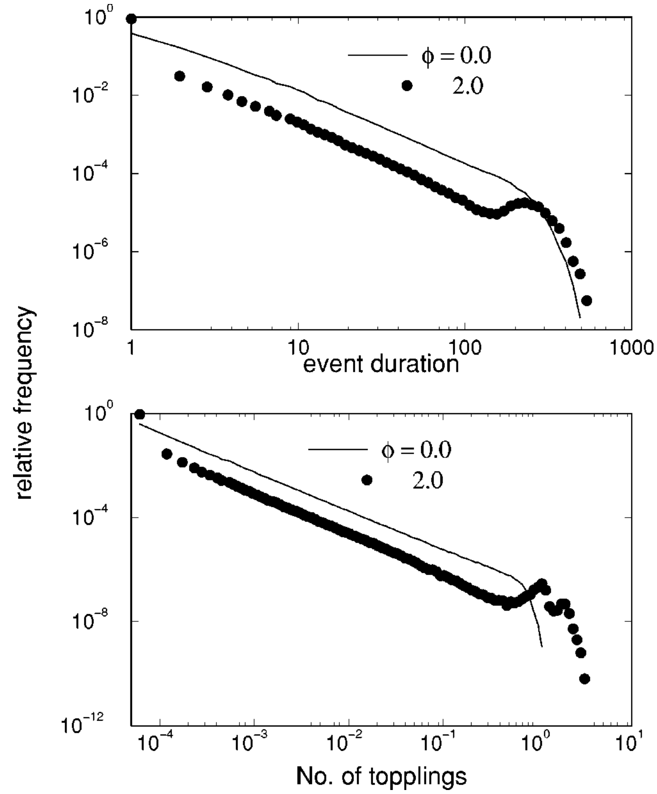

FIG. 2. The frequency distribution for the number of topplings and for the dimensionless event duration are shown for effective leakage factor $\phi = 2.0$; the plot with no leakage is shown for a comparison.

The upper part of this plot shows the distribution of event durations, defined in the model as the number of updates in the fast time scale that are required for relaxation. Again, the model with leakage reflects an excess of longer events over scaling: events with some range of large durations are more frequent, and the plot shows a local maximum close to its high end.

The SCCA models with no leakage dynamics show no signs of a characteristic-event regime. The power spectrum of the distribution of interevent times for large size events is white. The inclusion of leakage dynamics, however, changes this aspect radically, as shown in Fig. 3. The distribution of interevent times has a pronounced maximum, corresponding to a characteristic period between large rupture events. This feature is tuned by the leakage parameter, and will be more dominant as it increases.

The effects of the leakage dynamics on the statistical properties of the model's stress field are made more explicit in Fig. 4. This plot shows the time evolution, in the slow time scale, of its average value and roughness, together with the configuration entropy. This last quantity is a measure of the degree of ordering of the stress distribution [14]. Notice

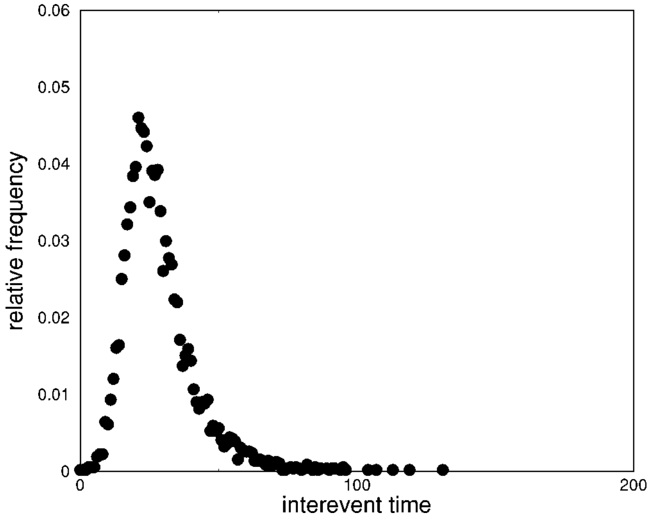

FIG. 3. The frequency distribution for dimensionless interevent times between large events that rupture a fraction of 0.9 of the total sites is shown for the dimensionless leakage factor $\phi = 2.0$. The plot shows the establishment of a characteristic event regime, superimposed on a complex background.

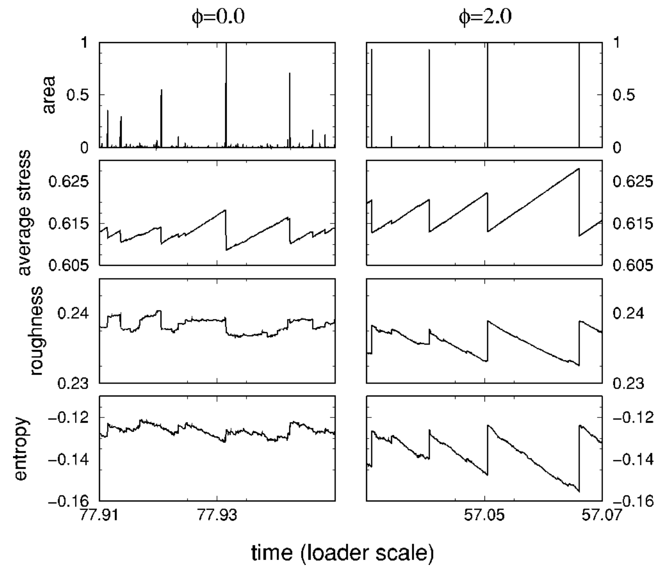

FIG. 4. The plot compares the time evolution of the dimensionless average stress, roughness, and configuration entropy for the model with no leakage ($\phi = 0$) and with leakage ($\phi = 2.0$). The upper plot shows the area of events in the time window selected, as a fraction of the lattice area. The model with leakage shows a much more regular dynamics and allows a larger buildup of the average stress, together with smaller roughness and entropy, which leads to a characteristic event regime.

that the average stress is an increasing function of the time between large events, both for the model without and with leakage, but that in the latter the overall average is higher. As opposed to what was seen in a SCCA model with no leakage but with varying dissipation and weakening of failed sites [14], we do not recognize a mode switching dynamics as present in our case. It remains to be seen what features would result from a combination of these dynamics.

We presented in this paper a SCCA model with aseismic creep superimposed on a seismic threshold dynamics, as recent laboratory experiments have shown to exist on solid-on-solid friction. In a mean-field approximation, with infinite range of uniform interactions, the equations for the inter-event stress time evolution can be solved. Computer simulations of the resulting SCCA model, combining the two dynamics, show a superposition of complex time and space patterns with a more regular occurrence of characteristic events, with rupture of the entire fault.

Research by J.S.S.M. was supported by CIRES, University of Colorado at Boulder; J.B.R. was supported by DOE (Grant No. DE-FG03-95ER14499), M.A. by DOE (Grant No. W-7405-ENG-36), and W.K. by DOE (Grant No. DE-FG02-95ER14498).

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